[3] Bannister, T. C., "Space Thermal Control Using Phase Change," TMX-53402, 1966, NASA.

[4] Pujado, P. R., Stermole, F. J., and Golden, J. O., "Melting of a Finite Paraffin Slab as Applied to Phase-Change Thermal Control," *Journal of Spacecraft and Rockets*, Vol. 6, No. 3, March 1969, pp. 280–284.

[5] Ehrlich, L. W., "A Numerical Method of Solving a Heat Flow Problem with Moving Boundary," *Journal of Association for Computing Machinery*, Vol. 5, No. 2, 1958, pp. 161–176.

[6] Grozka, P. G. and Fan, C., "Thermal Control by Freezing and Melting," Interim Rept., HREC-1123-1, LMSC/HRECA 791342, Contract NAS8-21123, 1968, Lockheed Missiles & Space Co., Huntsville, Ala.

[7] Muehlbauer, J. C. and Sunderland, J. E., "Heat Conduction with Freezing and Melting," *Applied Mechanics Review*, Vol. 18, No. 12, 1965, pp. 951–959.

[8] Ukanwa, A. O., *Phase Change Solidification Phenomena*, M.S. thesis, T-1289, Feb. 1970, Colorado School of Mines Golden, Colo.

Flow Coefficients for Supersonic Nozzles with Comparatively Small Radius of Curvature Throats

L. H. Back* and R. F. Cuffel†

Jet Propulsion Laboratory, Pasadena, Calif.

Nomenclature

a	=	sound speed
A	=	cross-sectional area
C_D	=	mass flow coefficient, $\dot{m}/\dot{m}_{1-D}$
$C_{D_{\text{inv}}}$	=	two-dimensional inviscid flow coefficient
D	=	diameter
$f(\gamma)$	=	$\{\gamma[2/(\gamma+1)]^{(\gamma+1)/(\gamma-1)}\}^{1/2}$
$\dot{m}$	=	mass flow rate
$\dot{m}_{1-D}$	=	one-dimensional value, $f(\gamma)p_tA_{th}/(RT_t)^{1/2}$
M	=	Mach number
p	=	pressure
r, r_{th}	=	radius and throat radius, respectively
r_c	=	throat radius of curvature
R	=	gas constant
$Re_{D_{th}}$	=	throat Reynolds number, $(\dot{m}D/A\mu^*)_{th}$
T	=	temperature
u	=	velocity
γ	=	specific heat ratio
δ^*	=	displacement thickness
λ,σ	=	divergent and convergent half-angles
μ,ρ	=	viscosity and density, respectively

Subscripts and superscripts

a	=	ambient back pressure
e	=	edge of boundary layer
i,t	=	inlet and stagnation conditions
th,w	=	throat and wall conditions
$(\ )^*$	=	sonic condition

Introduction

THIS Note is concerned with the determination of the mass flow rate through choked nozzles with emphasis on comparatively small radius of curvature throats. In the flow regime investigated (throat Reynolds numbers larger than 10^6) viscous (boundary layer) effects are not believed to be significant,[1] so that the flow field can be regarded as essentially isentropic. Mass flux nonuniformities for the air flows studies are then primarily caused by the throat configuration (Fig. 1)[2] and result in reduced mass flow rates below the ideal one-dimensional flow value, since in either the subsonic flow region near the centerline or the supersonic region near the wall the mass flux is less than at the sonic condition. The nozzles considered have circular-arc throats with values of the ratio of throat radius of curvature to throat radius r_c/r_{th} extending from 2 down to nearly 0, corresponding to a sharp-edged throat. Measured values of the flow coefficient C_D are presented for nozzles recently tested at the Jet Propulsion Laboratory (JPL) and for nozzles which have been previously tested in other investigations (Table 1). Of interest is the relative correspondence of the earlier measurements by Durham[3] that span a large range of r_c/r_{th} to the recent data since there is some question about their absolute magnitude because of the accuracy of the measurements that were made in a blow down facility. These measurements taken collectively provide a basis on which to evaluate the effect of r_c/r_{th} on the flow coefficient and to appraise existing and recently developed prediction methods for isentropic flow by other investigators.

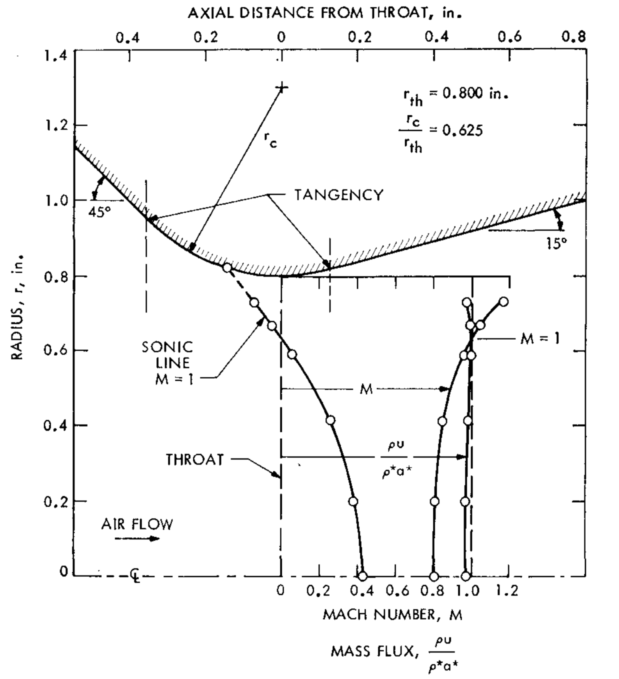


Fig. 1 Flow in the throat region of an axisymmetric nozzle. The Mach number and mass flux profiles are at the throat plane; the measurements are from Ref. 2 with air flow p_t = 70 psia, T_t = 540°R, $Re_{D_{th}}$ = 2.8 × 10⁶, adiabatic wall.

Present Tests

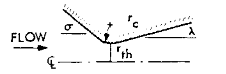
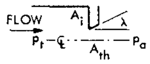
Tests were conducted in the auxiliary flow channel of the JPL hypersonic wind tunnel.[4] Air flowed steadily through a venturi meter, a plenum chamber, a contraction section, a constant-diameter duct, and the nozzle, into an evacuated chamber. Stagnation pressure measured with a pitot tube ranged from 25 to 100 psia. Stagnation temperature, measured with a thermocouple upstream of the nozzle inlet where the flow speed was low, was ~530°R. The nozzles tested had relatively steep convergent sections with convergent half-angles (σ) of 75° and 90° and with r_c/r_{th} = 0.25 and 0.49, respectively. Nozzles with these shapes are being considered for rocket engine applications because they are shorter

Received October 23, 1970; revision received December 3, 1970. This work presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract NAS 7-100, sponsored by NASA.

* Member Technical Staff, Propulsion Research and Advanced Concepts Section. Associate Fellow AIAA.

† Senior Engineer, Propulsion Research and Advanced Concepts Section. Member AIAA.

Table 1 Experimental data shown in Fig. 2

Configuration	Type	$\frac{A_{th}}{A_i}$	$\frac{p_a}{p_t}$	Source	Ref	$\frac{r_c}{r_{th}}$	r_{th}, in.	σ, deg	λ, deg	$Re_{D_{th}}$	$C_D = \frac{\dot{m}}{\dot{m}_{1D}}$
 AXISYMMETRIC CONVERGENT DIVERGENT NOZZLE				● PRESENT RESULTS, JPL		0.25 0.49	0.800 ↓	75 90	15 ↓	1.0×10^6 TO 4.0×10^6	0.951±0.005 0.969±0.005
				○ BACK, MASSIER AND GIER, JPL	6	0.625 2.0	0.800 0.902	45 30	15 ↓	1.8×10^6 TO 3.0×10^6	0.983±0.008 0.990±0.008
				□ NORTON AND SHELTON, JPL	5	0.35 0.55 0.75 1.0	0.563 ↓	30 ↓	15 ↓	5.9×10^6 TO 1.5×10^7	0.970±0.005 0.969±0.005 0.982±0.006 0.983±0.005
				△ DURHAM	3	0.04 0.125 0.50 1.0 2.0	0.500 ↓	90 ↓ 45	15 ↓	$\sim 10^7$ ↓	0.900 0.950 0.965 0.976 0.985
 SHARP EDGED ORIFICE	2-D SLIT	0.177	≅0.1	◇ BENSON AND POOL	14	→ 0	0.5	90	45	1.8×10^6	0.871±0.005
	CIRCULAR	0.25	0.03	◬ DURHAM △	15	→ 0 → 0	0.500 ↓	90 90	45 0	$\sim 10^7$ ↓	0.860 0.870
	CIRCULAR	≤0.015	0.15	▷ PERRY	16	→ 0	0.156 TO 0.250	90	31	3.3×10^5 TO 5.2×10^5	0.840
 SHARP THROAT CONICAL NOZZLE	$\frac{p_a}{p_t}$ DOWN TO 0.14			◊ THORNOCK ◇ ◇	18	→ 0 → 0 → 0	1.50	15 25 40	90 ↓	$\sim 7 \times 10^6$ at $\frac{p_a}{p_t} = 0.14$ (p_t=100 psia)	0.970 0.946 0.922

and lighter, with less surface area, and hence a lower total heat load. The entering flow was axial for the 75° nozzle. For the 90° nozzle, air flowed around a circular plate placed in the duct upstream to enter the nozzle radially.

Results

In Fig. 2, the solid symbols represent the present results and the other symbols are identified in Table 1. The data taken collectively indicate the magnitude of the decrease in C_D as r_c/r_{th} becomes smaller. The present values of C_D along with other measurements by Norton and Shelton[5] were obtained with the same venturi to measure the mass flow rate $\dot{m}$ and are believed to be accurate to within ± 0.005. The earlier measurements[6] that were obtained with wall cooling are less accurate, primarily because an orifice was used upstream to measure $\dot{m}$. The general agreement between the measurements at JPL and those by Durham[3] add credence to Durham's values. For these data, the convergent half-angle σ apparently has little influence on C_D, which is in accordance with Durham's results for various nozzles with σ ranging from

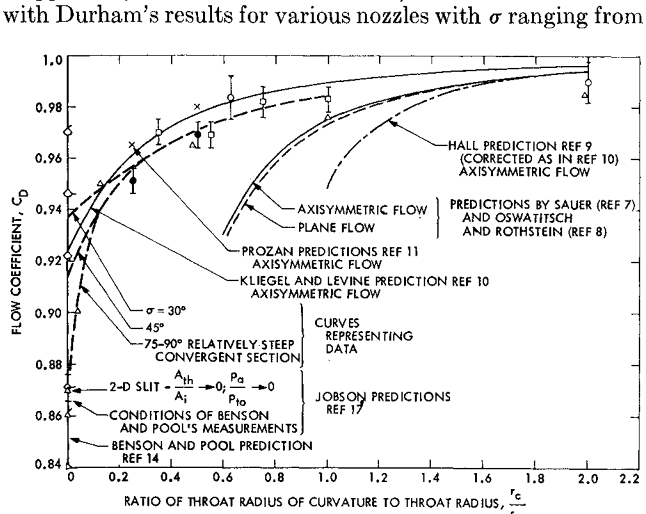


Fig. 2 Flow coefficients. Measurements are for air flows. Flow into the nozzles was axial except for the 90°–15° nozzle of the present investigation. All nozzle walls were essentially adiabatic except for the cooled nozzles T_w/T_t = 0.45–0.58 of Ref. 6. Predictions are for isentropic flow $\gamma = 1.4$.

10° to 90° but with more gradual throats r_c/r_{th} from 1 to 2 shown in Ref. 3.

The magnitude of the decrease in C_D found is considerably less than might be inferred from the earlier first-order approximate predictions of Sauer[7] and Oswatitsch and Rothstein[8] in which the velocity distribution was computed from the local configuration of the throat, i.e., r_c/r_{th}. It is also evident that Hall's approximate prediction,[9] in which three terms in the series expansion of the velocity components in inverse powers of r_c/r_{th} about the sonic condition were retained, leads to a poorer prediction than if only the first term in the series (Sauer prediction) were used. These predictions, however, are not expected to apply to nozzles with small r_c/r_{th} and are merely shown as reference curves.

Relatively good agreement with the measurements is provided by Kleigel and Levine's approximate prediction[10] that involves a series expansion similar to Hall, but in terms of inverse powers of $[1 + (r_c/r_{th})]$. However, for very small values of r_c/r_{th}, Kliegel and Levine's prediction, in which three terms in the series expansion were retained, appears to predict values of C_D that are too high for nozzles with relatively large convergent half-angles σ. In this approximate method the flow depends only upon $r_c/_{th}$ and not σ. Numerical solutions of the continuity and irrotational equations for steady flow through the nozzles presently tested, carried out by Prozan[11] using a relaxation technique (crosses in Fig. 2), also agree fairly well with the present measurements.

Attention is now focused on the relevance of measurements obtained with a sharp-edged orifice, i.e., $r_c/r_{th} \to 0$ and σ = 90° (Fig. 2) to the nozzle flow measurements. A sharp-edged differs from a conventional nozzle in that the flow is unable to negotiate the sharp corner and instead separates and becomes a jet which continues to contract downstream (vena contracta). For incompressible flow from a large reservoir, it is well known that C_D, which is the same as the contraction coefficient of the jet, i.e., $C_i = A_{vc}/A_{th}$ is $\sim$0.6. From inviscid flow calculations C_i is $\pi/(\pi + 2) = 0.611$ for a two-dimensional slit[12] and is slightly less, 0.591, for a circular orifice.[13] However for compressible flow, as the back pressure p_a decreases relative to the upstream pressure p_t, the contraction of the jet apparently becomes less and the mass flow rate continues to increase even though p_a decreases below the sonic pressure p^*. Consequently, the flow coefficient, defined in the usual way, increases. The measurements by Benson and Pool[14] for a two-dimensional slit and Durham[15] for a circular orifice at relatively low p_a/p_t (Fig. 2) indicate a value for C_D of about 0.87. The measurements by Perry,[16] at a somewhat higher p_a/p_t, indicate a slightly lower value for C_D. The predictions shown[14,17] agree reasonably well with the measurements.

It appears that the measured values of C_D for choked nozzles with comparatively small r_c/r_{th} and relatively large σ operated at relatively low p_a/p_t as they usually are, tend toward the sharp-edged orifice value as indicated in Fig. 2 by the curve faired through the data. However in the limiting case $r_c/r_{th} \to 0$, the measurements by Thornock[18] indicate a dependence of C_D on σ, with smaller σ's leading to higher C_D's (Fig. 2). This trend is in agreement with earlier measurements of Ref. 19 at larger p_a/p_t and predictions,[20] and does imply a dependence of C_D on σ for nozzles with very small r_c/r_{th}. For this situation, C_D might be estimated from the dashed curves in Fig. 2 for the σ's indicated. Little information is available on C_D for nozzles with relatively small σ and small r_c/r_{th}. It would appear that values of C_D for such nozzles would be higher.

Concluding Remarks

For supersonic nozzles with ratios of throat radius of curvature to throat radius between 0 and 2.0, tested at relatively high Reynolds numbers, the reduction in mass flow rate below the ideal one-dimensional flow value was caused primarily by the throat configuration rather than by boundary-layer effects. From the collection of data for air flows, the flow coefficient can be estimated (essentially the inviscid flow coefficient, $C_{D_{inv}}$) from the curves representing the data (Fig. 2). Consequently, information is available on how much the throat might be enlarged to accommodate the otherwise reduced mass flow rate or how much the chamber pressure might be increased. At lower Reynolds numbers where viscous (boundary-layer) effects become important, the actual flow coefficient might be calculated from the following relation that was derived in Ref. 1 by considering axisymmetric flow in the throat plane‡

$$C_D = C_{D_{\text{inv}}} - 2[(\overline{\rho u})_{\text{inv}}/(\rho u)^*](\delta^*/r_{th})[1 - (\delta^*/2r_{th})]$$

where $(\overline{\rho u}]_{\text{inv}}$ is the average value of the mass flux across the displacement thickness. To a first approximation, $(\overline{\rho u})_{\text{inv}}$ might be evaluated at the wall for an inviscid flow.

References

[1] Massier, P. F., "Viscous Effects on the Flow Coefficient for a Supersonic Nozzle," *AIAA Journal*, Vol. 8, No. 3, March 1970, pp. 605–607.

[2] Cuffel, R. F., Back, L. H., and Massier, P. F., "Transonic Flow Field in a Supersonic Nozzle with Small Throat Radius of Curvature," *AIAA Journal*, Vol. 7, No. 7, July 1969, pp. 1364–1366.

[3] Durham, F. P., "The Performance Characteristics of Small Rocket-Type Nozzle," Summary Rept. 6, Contract N-123S-82337, T.O. 1, June 1955, Univ. of Colorado, Boulder, Colo.

[4] Strand, L., Newton, J. F., Jr., and Herrera, J. G., "Initial Performance of a New Nozzle Gas-Dynamics Test Facility," Tech. Memo 33-310, Nov. 1966, Jet Propulsion Lab., Pasadena, Calif.

[5] Norton, D. and Shelton, S., "Performance of Rocket Nozzles with Low Radius of Curvature Ratios," *JPL Space Programs Summary No. 37-55*, Vol. III, Feb. 1969, Pasadena, Calif., pp. 167–169.

[6] Back, L. H., Massier, P. F., and Gier, H. L., "Comparison of Measured and Predicted Flows through Conical Supersonic Nozzles, with Emphasis on the Transonic Region," *AIAA Journal*, Vol. 3, No. 9, Sept. 1965, pp. 1606–1614.

[7] Sauer, R., "General Characteristics of the Flow Through Nozzles at Near Critical Speeds," TM-1147, June 1947, NACA.

[8] Oswatitsch, K. and Rothstein, W., "Flow Pattern in a Converging-Diverging Nozzle," TM-1215, March 1949, NACA.

‡ For two-dimensional flow through a nozzle with a rectangular cross section, $C_D = C_{D_{\text{inv}}} - [(\overline{\rho u})_{\text{inv}}/(\rho u)^*](\delta^*/h_{th})$; where h_{th} is the throat half-height.

[9] Hall, I. M., "Transonic Flow in Two-Dimensional and Axially-Symmetric Nozzles," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. XV, Pt. 4, 1962, pp. 487–508.

[10] Kleigel, J. R. and Levine, J. N., "Transonic Flow in Small Throat Radius of Curvature Nozzles," *AIAA Journal*, Vol. 7, No. 7, July 1969, pp. 1375–1378.

[11] Prozan, R. J., "Transonic Flow in a Converging-Diverging Nozzle," LMSC/HREC D162177, Contract NAS7-743, 1970, Lockheed Missiles & Space Co., Huntsville, Ala.

[12] Milne-Thomson, *Theoretical Hydrodynamics*, Macmillan, New York, 1968, p. 310.

[13] Bloch, E., "Numerical Solution of Free Boundary Problems by the Method of Steepest Descent," *The Physics of Fluids Supplement II, High Speed Computing in Fluid Dynamics*, Dec. 1969, pp. 129–132.

[14] Benson, R. S. and Pool, D. E., "The Compressible Flow Discharge Coefficients for a Two-Dimensional Slit," *International Journal of Mechanical Sciences*, Vol. 7, May 1965, pp. 337–353.

[15] Durham, F. P., "The Performance Characteristics of Small Rocket-Type Nozzles," Rept. 2, Contract N-123S-82337, T.O. 1, June 1953, Univ. of Colorado, Boulder, Colo.

[16] Perry, J. A., Jr., "Critical Flow Through Sharp-Edged Orifices," *Transactions of ASME*, Vol. 71, 1949, pp. 757–764.

[17] Jobson, D. A., "On the Flow of a Compressible Fluid through Orifices," *Proceedings Institution of Mechanical Engineers*, Vol. 169, 1955, pp. 767–772.

[18] Thornock, R. L., "Experimental Investigation of the Flow Through Convergent Conical Nozzles," Document D6-20375, Sept. 1968, The Boeing Co., Seattle, Wash.

[19] Grey, R. E. and Wilsted, H. D., "Performance of Conical Jet Nozzles in Terms of Flow and Velocity Coefficients," TN-1757, Nov. 1948, NACA.

[20] Wehofer, S. and Moger, W. C., "Transonic Flow in Conical Convergent and Convergent-Divergent Nozzles with Non-uniform Inlet Conditions," AIAA Paper 70-635, San Diego, Calif., 1970.

Nonlinear Proportional Navigation and the Minimum Time-to-Turn

A. G. RAWLING*
General Electric Company, King of Prussia, Pa.

Nomenclature

A	= azimuth angle of the velocity vector
$B, \ldots, E$	= coefficient functions in the quartic Eq. (23)
g	= acceleration due to gravity
k	= Ng, missile lateral acceleration (N is a signed number)
LOS	= line of sight between missile and target
R	= missile-target separation range along the LOS
t,T	= time and its reciprocal ($T = t^{-1}$), respectively
V	= flight speed
x,y	= inertially-fixed Cartesian coordinate axes in the plane
$\alpha,\beta,\gamma,\epsilon$	= coefficients in the biquadratic Eq. (29)
Δ,δ	= function differences in Eqs. (11–12, 18–21)
ΔHE	= missile heading error at $t = 0$
σ	= LOS angle measured from the x-axis (Fig. 1)
τ	= square of reciprocal time T in Eq. (33), ($\tau = T^2 = (t^{-2})$

Received August 16, 1970; revision received October 26, 1970. The author is indebted to T. J. McFall for assistance in verifying the method.

* Consulting Engineer, Systems Management Laboratory, Re-Entry and Environmental Division. Member AIAA.